On Foucault's Pendulum.

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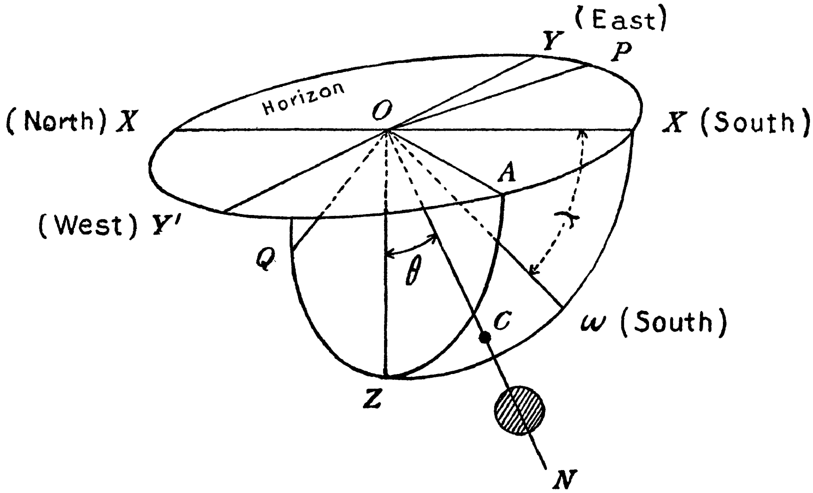
In this paper we shall consider the motion of a physical pendulum on the surface of the earth, taking into account the rotation of the earth about its axis. The initial velocity of the pendulum relatively to the earth will be supposed equal to zero, as in the famous experiment of Foucault with his practically mathematical pendulum. The name of "Foucault's pendulum" is therefore retained, although oscillations of any finite amplitude will be considered.*



Let O be a point on the surface of the earth; OXZ the meridian of this point; $O\omega$ the direction of the axis of the earth towards the south. Then $\angle XO\omega$ is the latitude λ of the place, OX being the intersection of the meridian plane with the horizon XOY, directed again towards the south; OY in the plane of the horizon 90° from OX and towards the east; OZ in the direction of the force of gravity. Farther, let ON be a physical pendulum, θ the angle its axis makes with OZ, the pendulum being a body of rotation. Let OP, OQ, ON be a system of three principal axes of inertia about the point O, the axis ON coinciding with the geometrical axis of the pendulum, OQ under 90° to ON in the plane $OANZ$

*As is well known, only very small oscillations were given to Foucault's pendulum.

(the plane which would be the plane of oscillations of the pendulum but for the disturbance due to the rotation of the earth); OP, perpendicular to this plane in the sense shown on the figure. Let farther C be the centre of inertia of the pendulum and $OC = l$; M be the mass of the body; A, A, C be the three principal moments of inertia of the pendulum about the point O; $\gamma_1, \gamma_2, \gamma_3$ the angles which the axes OP, OQ, ON make with the direction of the axis (ω) of the earth. Finally, let θ, ϕ, ψ be the three Eulerian angles which define the position of the pendulum relative to the earth, i. e. $\theta = \angle NOZ$; $\phi = \angle POX$; ψ = angle of a determinate plane passing through ON and fixed in the body, with the plane OZN. Then we shall have

$$\begin{aligned} \cos\gamma_1 &= \cos\lambda\cos\phi, \\ \cos\gamma_2 &= \sin\lambda\sin\theta - \cos\lambda\cos\theta\sin\phi, \\ \cos\gamma_3 &= \sin\lambda\cos\theta + \cos\lambda\sin\theta\sin\phi. \end{aligned} \tag{1}$$

We shall make use of Bour's differential equations for the relative motion

$$\frac{d}{dt}\left(\frac{\partial T_2}{\partial q_k'}\right) - \frac{\partial T_2}{\partial q_k} = \frac{\partial (U+K)}{\partial q_k},$$

where T_2 is the kinetic energy of the *absolute* rotation, $U+K$ is the potential of the force of *gravity* (and not of attraction)* in the case of the earth. Hence

$$U + K = Mgl\cos\theta, \tag{2}$$

$$T_2 = \tfrac{1}{2}\{A[(\theta' + \omega\cos\gamma_1)^2 + (\phi'\sin\theta + \omega\cos\gamma_2)^2] + C(r + \omega\cos\gamma_3)^2\}, \tag{3}$$

where
$$r = \psi' + \phi'\cos\theta,$$

and ω is the angular velocity of the rotation of the earth. Bour's equation gives first ($q_k = \psi$),

$$\frac{d}{dt}\left(\frac{\partial T_2}{\partial \psi'}\right) = 0,$$

as neither T_2 nor $U+K$ contains the variable ψ, and thus we have a first integral

$$r + \omega\cos\gamma_3 = \psi' + \phi'\cos\theta + \omega\cos\gamma_3 = c_1.$$

We have supposed the initial velocity of the pendulum equal to zero, hence $c_1 = \omega\cos\gamma_{30}$ and

$$r + \omega\cos\gamma_3 = \omega\cos\gamma_{30}, \tag{5}$$

* U is the potential of the force of attraction.

where $\cos\gamma_{30}$ means the substitution of the initial values θ_0, ϕ_0 into the expression of $\cos\gamma_3$.

Next take $q_k = \phi$: $$\frac{d}{dt}\left(\frac{\partial T_2}{\partial\phi'}\right) - \frac{\partial T_2}{\partial\phi} = 0,$$

or

$$\frac{d}{dt}\left[A(\phi'\sin\theta + \omega\cos\gamma_2)\sin\theta + C\omega\cos\gamma_{30}\cos\theta\right] = -A\omega\cos\lambda\sin\theta\left[\theta' + \phi'\cos\theta\cos\phi\right],$$

leaving off the terms of the order of ω^2.* After a slight transformation this equation becomes

$$\frac{d}{dt}\left[A\sin^2\theta(\phi' + \omega\sin\lambda) + C\omega\cos\gamma_{30}\cos\theta\right] = -2A\omega\cos\lambda\sin^2\theta\sin\phi\theta'. \tag{6}$$

If we neglect terms of the order of ω^2 and higher, we may substitute in the right-hand member of (6) the value of $\sin^2\theta\sin\phi.\theta'$ calculated in the supposition of the immobility of the earth, because this expression has the factor ω. Let ϑ be the angle formed by the axis of the pendulum with OZ in this hypothesis. Then we may instead of $\sin^2\theta\sin\phi.\theta'$ in (6) substitute $\sin\phi_0\sin^2\vartheta.\vartheta'$. Integrating the equation after that and putting for brevity

$$f_1(\theta) = \sin^2\theta_0 - \sin^2\theta + \frac{C}{A}\cos\theta_0(\cos\theta_0 - \cos\theta), \tag{7$_1$}$$

$$f_2(\theta) = \frac{C}{A}\sin\theta_0(\cos\theta_0 - \cos\theta), \tag{7$_2$}$$

$$f_3(\vartheta) = \theta_0 - \sin\theta_0\cos\theta_0 - (\vartheta - \sin\vartheta\cos\vartheta) \tag{7$_3$}$$

we shall find

$$\sin\theta.\phi' = \omega\frac{f_1(\theta)\sin\lambda + [f_2(\theta) + f_3(\vartheta)]\cos\lambda\sin\phi_0}{\sin\theta}. \tag{8}$$

This value of $\sin\theta.\phi'$ we now substitute in the third and last integral, that of the kinetic energy in the relative motion:

$$A(\theta'^2 + \sin^2\theta.\phi'^2) = 2Mgl(\cos\theta - \cos\theta_0). \tag{9}$$

Now, the expression (8) shows that $\sin^2\theta.\phi'^2$ involves the factor ω^2. But this term *cannot be neglected.* This exception is due to the fact that in the right-hand

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*In this problem, like in all other problems of motions on the surface of the earth, it is useless to keep terms of the order of ω^2, if the force of gravity is considered as constant.

member of (8) $\sin\theta$ appears in the denominator. We do not know *a priori* what the minimum value of θ may be. It may perhaps be zero, like the minimum value of ϑ (i. e. like in the ordinary pendulum), in which case the right-hand member of (8) would become infinite; or it may be of the order of ω, in which case the same expression would have a finite value. It follows from this remark that the term $\frac{\omega^2}{\sin^2\theta}\{f_1(\theta)\sin\lambda+[f_2(\theta)+f_3(\vartheta)]\cos\lambda\sin\phi_0\}^2$ *cannot be neglected a priori whatever be the desired approximation.* There are, however, some terms in the expression (8) which may be dropped, namely, $-\sin^2\theta$ in $(7)_1$, $(\vartheta-\sin\vartheta\cos\vartheta)$ in $(7)_3$. This is obvious. Hence, we must substitute for $\sin^2\theta.\phi'^2$ into the left-hand member of (9) the expression $\frac{\omega^2 f^2(\theta)}{\sin^2\theta}$, where

$$f(\theta)=\sin\lambda\left[\sin^2\theta_0+\frac{C}{A}\cos\theta_0(\cos\theta_0-\cos\theta)\right]$$
$$+\cos\lambda\sin\phi_0\left[\frac{C}{A}\sin\theta_0(\cos\theta_0-\cos\theta)+\theta_0-\sin\theta_0\cos\theta_0\right]. \quad (10)$$

We then obtain the equation

$$\left(\frac{d\cos\theta}{dt}\right)^2=\frac{2Mgl}{A}\sin^2\theta(\cos\theta-\cos\theta_0)-\omega^2 f^2(\theta), \quad (11)$$

the integration of which gives

$$\cos\theta=\cos\theta_0\,\mathrm{sn}^2\mu(t+T)+(1-\tfrac{1}{2}\varepsilon^2)\,\mathrm{cn}^2\mu(t+T),$$

where

$$\left.\begin{aligned}\mu&=\sqrt{\frac{Mgl}{A}},\\ T&=\sqrt{\frac{A}{Mgl}}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}},\\ k&=\sin\frac{\theta_0}{2},\end{aligned}\right\} \quad (12)$$

$$\varepsilon=\omega\frac{f(0)}{2k\mu}. \quad (13)$$

We may write $\cos\varepsilon$ instead of $(1-\frac{1}{2}\varepsilon^2)$, then

$$\cos\theta=\cos\theta_0\,\mathrm{sn}^2\mu(t+T)+\cos\varepsilon\,\mathrm{cn}^2\mu(t+T). \quad (14)$$

To determine ϕ, the equation (8) gives

$$\phi'=-\omega\sin\lambda+\omega\frac{f(\theta)}{\sin^2\theta}-\omega\cos\lambda\sin\phi_0\frac{\vartheta-\sin\vartheta\cos\vartheta}{\sin^2\theta}. \quad (15)$$

The expression of ϑ in function of time is found from (14), where we only need to put $\varepsilon = 0$. Then $\sin\frac{\vartheta}{2} = \sin\frac{\theta_0}{2}\,\mathrm{sn}\,\mu(t+T)$ and $\cos\frac{\vartheta}{2} = \mathrm{dn}\,\mu(t+T)$. Thus $\frac{\vartheta - \sin\vartheta\cos\vartheta}{\sin^2\theta}$ is readily obtained in function of t. Let us put

$$\phi(t) = \frac{\vartheta - \sin\vartheta\cos\vartheta}{\sin^2\theta} \text{ and} \int_0^t \phi(t)\,dt = \Phi(t). \tag{16}$$

Then, as easily seen, $\Phi(t)$ is a periodical function of the period $2T$. We have furthermore

$$\frac{f(\theta)}{\sin^2\theta} = \frac{\frac{1}{2}f(0)}{1-\cos\theta} + \frac{\frac{1}{2}f(\pi)}{1+\cos\theta},$$

$$1 - \cos\theta = -2k^2(\mathrm{sn}^2\eta i - \mathrm{sn}^2\mu(t+T)),$$

$$1 + \cos\theta = \quad 2\,\mathrm{dn}^2\mu(t+T),$$

where we have put $\eta = \frac{\varepsilon}{2k}$; and very simple integrations give

$$\int_0^t \frac{dt}{\mathrm{sn}^2\eta i - \mathrm{sn}^2\mu(t+T)} = \frac{i}{2\mu\eta}\lg\frac{\Theta_2(\mu t - \eta i)}{\Theta_2(\mu t + \eta i)} - \zeta t,$$

$$\int_0^t \frac{dt}{\mathrm{dn}^2\mu(t+T)} = \frac{1}{k'^2}(1-\zeta)t + \frac{1}{\mu k'^2}\,\frac{\Theta'(\mu t)}{\Theta(\mu t)}.$$

A transformation, which will be easily verified by the reader, gives

$$\frac{1}{2i}\lg\frac{\Theta_2(\mu t - \eta i)}{\Theta_2(\mu t + \eta i)} = \mathrm{tn}^{-1}\left[\eta\,\frac{\Theta_2'(\mu t)}{\Theta_2(\mu t)}\right].$$

In these formulas, as is usual,

$$\zeta = \frac{\Theta''(0)}{\Theta(0)} = \frac{1}{\mu T}\int_0^{\mu T} k^2\,\mathrm{sn}^2 x\,dx. \tag{17}$$

Substituting the above formulas into (15), integrating and putting for the sake of brevity

$$N = -\sin\lambda + \tfrac{1}{4}\left[\frac{f(0)}{k^2}\zeta + \frac{f(\pi)}{k'^2}(1-\zeta)\right], \tag{18}$$

$$F(t) = \frac{f(\pi)}{4k'^2\mu}\,\frac{\Theta'(\mu t)}{\Theta(\mu t)} - \cos\lambda\sin\phi_0\,\Phi(t), \tag{19}$$

we shall find

$$\phi = \phi_0 + \omega N t + \operatorname{tn}^{-1}\left[\eta \frac{\Theta_2'(\mu t)}{\Theta_2(\mu t)}\right] + \omega F(t). \tag{20}$$

If we put furthermore

$$\phi_1 = \operatorname{tn}^{-1}\left[\eta \frac{\Theta_2'(\mu t)}{\Theta_2(\mu t)}\right] + \omega F(t),$$

or

$$\operatorname{tg}\left[\phi_1 - \omega F(t)\right] = \eta \frac{\Theta_2'(\mu t)}{\Theta_2(\mu t)}, \tag{21}$$

the motion of the axis of the pendulum may be represented in the following way:

1. Equation (14) shows that *the axis of the pendulum oscillates between the positions $\theta = \theta_0$ and $\theta = \varepsilon$, never passing through the vertical.*

2. *The equations* (14) *and* (21) *represent a closed cone with a plane of symmetry which would be the plane of oscillations but for the disturbance due to the rotation of the earth.*

3. *If we rotate the cone just defined, about the vertical OZ with the constant angular velocity ωN; the combined motion: of the cone about OZ, and of the axis of the pendulum on this cone, will represent the motion of the axis of the pendulum relatively to the earth.*

4. *The rotation of the cone defined above may take place towards the west or towards the east, or the cone may be fixed relatively to the earth, according to whether N is less than, greater than, or equal to zero.**

In fact, formula (18) can be written in the following way:

$$N = -n_1 \sin\lambda + n_2 \cos\lambda \sin\phi_0, \tag{22}$$

$$n_1 = \zeta k^2 + (1-\zeta) k'^2 - \frac{C}{2A}(1-2\zeta)\cos\theta_0, \tag{23}$$

$$n_2 = \frac{\theta_0 - \sin\theta_0\cos\theta_0}{\sin^2\theta_0}\left[\zeta k'^2 + (1-\zeta)k^2\right] + \frac{C}{2A}(1-2\zeta)\sin\theta_0, \tag{24}$$

and it is easily verified that *both n_1 and n_2 have always positive values.* Hence N is $\gtreqless 0$ according as

$$n_2 \sin\phi_0 \gtreqless n_1 \operatorname{tg}\lambda. \tag{25}$$

It follows from this discussion that *the rotation of the cone* defined above, *at the same latitude, depends* 1) *on the amplitude of the oscillations*; 2) *on the construction*

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*See Comte de Sparre: Sur le mouvement du pendule conique à la surface de la terre. (Thèse de Doctorat.)

of the pendulum, and 3) *on its initial orientation.* The dependence on the amplitude and the orientation has been indicated first by Count de Sparre in his doctor thesis, in which, however, only the motion of a *mathematical* pendulum was considered. The influence of the construction of the pendulum on the results, as seen from the formulas developed in this paper, offers a greater field for experiments which would be very interesting.

Formula (22) shows that, *ceteris paribus, the absolute value of N is maximum or minimum, if the pendulum is started in the plane of the meridian; maximum if initially deviated towards the north; minimum if towards the south.*

$$|N|_{\text{max.}} = n_1 \sin\lambda + n_2 \cos\lambda,$$
$$|N|_{\text{min.}} = |n_1 \sin\lambda - n_2 \cos\lambda|.$$

In order to complete the solution it remains to determine the angle ψ in function of t. This is done very easily, but the limits of this paper do not allow the reproduction of the calculation here.

To conclude, let us see what the above formulas become in the case of very small amplitudes of oscillations, neglecting powers of θ_0 higher than the second.

Formula (21) becomes

$$\operatorname{tg}\phi_1 = \eta\,\frac{\Theta_2'(\mu t)}{\Theta_2(\mu t)} = -\frac{\varepsilon}{\theta_0}\operatorname{tg}\mu t, \tag{21$'$}$$

and instead of the equation (14) we may write

$$\theta^2 = \theta_0^2\cos^2\mu t + \varepsilon^2\sin^2\mu t. \tag{14$'$}$$

The elimination of t between the equations (21)$'$ and (14)$'$ gives the elliptic cone

$$\frac{1}{\theta^2} = \frac{\cos^2\phi_1}{\theta_0^2} + \frac{\sin^2\phi_1}{\varepsilon^2}. \tag{26}$$

The intersection of this cone with a horizontal plane at the distance $\sqrt{A}$ from the point O and fixed with regard to the cone (26), gives an ellipse with the semiaxes

$$a = \sqrt{A}\,\theta_0,$$
$$b = \sqrt{A}\,\varepsilon = \frac{2A - C}{2A}\sqrt{\frac{A}{Mgl}}\,a\omega\sin\lambda.$$

A first approximation gives for N the value $-\frac{2A-C}{2A}\sin\lambda$. These results are in perfect accordance with those of Mr. Kamerlingh Onnes.*

A more accurate value of N is obtained by retaining second powers of θ_0. Formula (22) then gives

$$N=-\frac{2A-C}{2A}\sin\lambda+\frac{C}{2A}\theta_0\cos\lambda\sin\phi_0+\frac{3}{8}\frac{A-C}{A}\theta_0^2\sin\lambda,$$

which differs from the result obtained by Mr. Kamerlingh Onnes, who gives the value

$$N=-\frac{2A-C}{2A}\sin\lambda+\frac{C}{2A}\theta_0\cos\lambda\sin\phi_0+\frac{3}{8}\frac{2A-C}{2A}\theta_0^2\sin\lambda.$$

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† Over de betrekkelijke Beweging. Nieuw Archief voor Wiskunde, Deel V, p. 162.